Non-collapsing quasiparticle random phase approximation for nuclear double-beta decay*

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Abstract

We show how the longstanding problem of the collapse of the charge-exchange QRPA near the physical value of the force strength can be circumvented. This is done by including the effect of ground state correlations into the QRPA equations of motion. The corresponding formalism, called renormalized QRPA, is briefly outlined and its consequences are discussed in the framework of a schematic model for the two-neutrino double beta decay in the $^{100}Mo \rightarrow {}^{100}Ru$ system. The question of the conservation of the Ikeda sum rule is also addressed within the new formalism.

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Double beta ($\beta\beta$) decays occur in medium-mass nuclei that are rather far from closed shells. The nuclear structure method most widely used in the evaluation of $\beta\beta$ rates for two-neutrino decay mode ($\beta\beta_{2\nu}$) as well as for the neutrinoless mode ($\beta\beta_{0\nu}$) is therefore the quasiparticle random phase approximation (QRPA) [1]. These calculations, in which the $\beta\beta_{2\nu}$ matrix elements $\mathcal{M}_{2\nu}$ are approximated by their $J^\pi = 1^+$ component (*i.e.,* $\mathcal{M}_{2\nu} \cong \mathcal{M}_{2\nu}(J^\pi = 1^+)$), explain the smallness of the measured transition rates. [1] However, the actual value of $\mathcal{M}_{2\nu}$ depends sensitively on the strength g^{pp} of the particle-particle force in the $S = 1$, $T = 0$ channel. For realistic forces of finite range $\mathcal{M}_{2\nu}$ passes through zero near $g^{pp} = 1$ *i.e.,* near the physical value of this coupling constant. This feature makes the actual value of $\mathcal{M}_{2\nu}$ rather uncertain. What is still more distressing, is that QRPA collapses for $g^{pp} \gtrsim 1$. One may thus suspect that $\mathcal{M}_{2\nu}$ goes through zero simply because the approximation breaks up. In other words, the smallness of $\mathcal{M}_{2\nu}$ in the QRPA could be just an artifact of the model. (One should remember that in the Tamm-Dancoff approximation, *i.e.,* in the absence of the ground state correlations, $\mathcal{M}_{2\nu}$ always increases with g^{pp}.) Yet, it has been pointed out more than once that the zero of $\mathcal{M}_{2\nu}$ is not engendered by the collapse of the QRPA, but arises instead from the partial restoration of the $SU(4)$ Wigner symmetry [3].

It has been shown recently that within the QRPA the above behavior of the 2ν amplitude can be summarized as [4]

$$\mathcal{M}_{2\nu} \cong \mathcal{M}_{2\nu}(g^{pp} = 0)\frac{1 - g^{pp}/g_0^{pp}}{1 - g^{pp}/g_1^{pp}}, \quad \text{with} \quad g_0^{pp} \cong 1, g_1^{pp} \gtrsim g_0^{pp}, \tag{1}$$

where g_0^{pp} and g_1^{pp} denote respectively the zero and the pole of $\mathcal{M}_{2\nu}$. Moreover, it has been suggested that within the QRPA the 0ν amplitude behaves as

$$\begin{aligned}\mathcal{M}_{0\nu} \quad &\cong \quad \mathcal{M}_{0\nu}(J^\pi = 1^+; g^{pp} = 0)\frac{1 - g^{pp}/g_0^{pp}}{\sqrt{1 - g^{pp}/g_1^{pp}}} \\ &+ \quad \mathcal{M}_{0\nu}(J^\pi \neq 1^+; g^{pp} = 0)(1 - g^{pp}/g_2^{pp}),\end{aligned} \tag{2}$$

where $g_2^{pp} \gg g_1^{pp}$ [4]. This means that the $J^\pi = 1^+$ component of $\mathcal{M}_{0\nu}$ exhibits the zero and the pole at the same value of g^{pp} as $\mathcal{M}_{2\nu}$. Thus, the theoretical estimation of $\mathcal{M}_{0\nu}$, and therefore the determination of the limit for the effective neutrino mass $< m_\nu >$, is also uncertain as that of $\mathcal{M}_{2\nu}$.

Several modifications of the QRPA have been proposed in order to change the above behavior in a qualitative way, including higher order RPA corrections [5], nuclear deformation [6], single-particle self-energy BCS terms [7] and particle number projection [8].

[1] It was found that the contributions of the odd-parity nuclear operators to the $\beta\beta_{2\nu}$-decay are significant when compared with the experimental data [2].

Yet, none of these amendments inhibits the collapse of the charge-exchange QRPA. In the present work we show that this can be achieved by including the effect of ground state correlations in the QRPA equations of motion. The corresponding formalism, referred to as renormalized QRPA (RQRPA), was originally introduced by Rowe [9]. It has been used recently by Catara *et al.,* [10, 11] in the evaluation of the charge transition densities and properties of the charge-conserving collective states. We briefly outline below the RQRPA formalism for charge-exchange excitations, and discuss it within a schematic model for $\mathcal{M}_{2\nu}$.

We begin by defining excited states $|\lambda J\rangle$ that are built by the action of the charge-exchange operators

$$\Omega^{\dagger}(\lambda J) = \sum_{pn} \left[X_{pn}(\lambda J) A^{\dagger}_{pn}(J) - Y_{pn}(\lambda J) A_{pn}(\bar{J}) \right], \tag{3}$$

on the correlated ground state $|0\rangle$. Here $A^{\dagger}_{pn}(J) = [\alpha^{\dagger}_p \alpha^{\dagger}_n]^J$, and $\alpha^{\dagger}_p$ and $\alpha^{\dagger}_n$ are quasiparticle creation operators for protons and neutrons. The amplitudes X and Y, the eigenvalues ω_λ and $|0\rangle$ are obtained from the equations of motion (EM)

$$\langle 0| \left[\delta\Omega(\lambda\bar{J}), H, \Omega^{\dagger}(\lambda J) \right]^0 |0\rangle = \omega_\lambda \langle 0| \left[\delta\Omega(\lambda\bar{J}), \Omega^{\dagger}(\lambda J) \right]^0 |0\rangle, \tag{4}$$

with the condition

$$\Omega(\lambda J)|0\rangle = 0, \text{for all } \lambda, J. \tag{5}$$

The usual QRPA equations result from (4) when $|0\rangle$ is approximated by the BCS ground state $|BCS\rangle$ and (5) is ignored. In the RQRPA one takes the ground state correlations (GSC) introduced by (5) in the EM (4) partially into account. First note that we have now

$$\hat{J}^{-1} \langle 0| \left[A_{pn}(\bar{J}), A^{\dagger}_{p'n'}(J') \right]^0 |0\rangle = \delta_{pp'} \delta_{nn'} \delta_{JJ'} D_{pn}, \tag{6}$$

with

$$D_{pn} = \hat{J}^{-1} \langle 0| \left[A_{pn}(\bar{J}), A^{\dagger}_{pn}(J) \right]^0 |0\rangle = 1 - \mathcal{N}_p - \mathcal{N}_n, \tag{7}$$

where $\hat{J} \equiv \sqrt{2J+1}$ and $\mathcal{N}_p$ ($\mathcal{N}_n$) are the proton (neutron) quasiparticle occupations

$$\mathcal{N}_t = \hat{j}_t^{-1} \langle 0| [\alpha^{\dagger}_t \alpha_{\bar{t}}]^0 |0\rangle. \tag{8}$$

The label t stands for p and n.

We define next "renormalized" two quasiparticle operators as

$$\mathcal{A}^{\dagger}_{pn}(J) = A^{\dagger}_{pn}(J) D^{-1/2}_{pn}, \tag{9}$$

which satisfy the relation

$$\hat{J}^{-1} \langle 0| \left[\mathcal{A}_{pn}(\bar{J}), \mathcal{A}^{\dagger}_{p'n'}(J') \right]^0 |0\rangle = \delta_{pp'} \delta_{nn'} \delta_{JJ'}. \tag{10}$$

The crucial RQRPA assumption is the generalized quasiboson approximation

$$\hat{J}^{-1}\left[\mathcal{A}_{pn}(\bar{J}),\mathcal{A}^{\dagger}_{p'n'}(J')\right]^{0}\cong\hat{J}^{-1}\langle 0|\left[\mathcal{A}_{pn}(\bar{J}),\mathcal{A}^{\dagger}_{p'n'}(J')\right]^{0}|0\rangle=\delta_{pp'}\delta_{nn'}\delta_{JJ'}, \tag{11}$$

The RQRPA equations follow straightforwardly after replacing $A^{\dagger}_{pn}(J)$ by $\mathcal{A}^{\dagger}_{pn}(J)$ in the expression for $\Omega^{\dagger}(J)$ and using (11) in the EM (4). We get in this way [9, 11]

$$\left(\begin{array}{cc}\mathsf{A}(J) & \mathsf{B}(J)\\ \mathsf{B}^{*}(J) & \mathsf{A}^{*}(J)\end{array}\right)\left(\begin{array}{c}\mathsf{X}(\lambda J)\\ \mathsf{Y}(\lambda J)\end{array}\right)=\omega_{\lambda J}\left(\begin{array}{c}\mathsf{X}(\lambda J)\\ -\mathsf{Y}(\lambda J)\end{array}\right), \tag{12}$$

where

$$\mathsf{X}_{pn}(\lambda J)\equiv X_{pn}(\lambda J)D^{1/2}_{pn}\quad\text{and}\quad\mathsf{Y}_{pn}(\lambda J)\equiv Y_{pn}(\lambda J)D^{1/2}_{pn}, \tag{13}$$

are the renormalized amplitudes. The submatrices $\mathsf{A}(J)$ and $\mathsf{B}(J)$ are found as

$$\begin{aligned}\mathsf{A}_{pn,p'n'}(J) &= (\epsilon_p+\epsilon_n)\delta_{pp'}\delta_{nn'}+D^{1/2}_{pn}\left[F(pn,p'n',J)(u_pv_nu_{p'}v_{n'}+v_pu_nv_{p'}u_{n'})\right.\\ &+ \left.G(pn,p'n',J)(u_pu_nu_{p'}u_{n'}+v_pv_nv_{p'}v_{n'})\right]D^{1/2}_{p'n'},\\ \mathsf{B}_{pn,p'n'}(J) &= D^{1/2}_{pn}\left[F(pn,p'n',J)(v_pu_nu_{p'}v_{n'}+u_pv_nv_{p'}u_{n'})\right.\\ &- \left.G(pn,p'n',J)(u_pu_nv_{p'}v_{n'}+v_pv_nu_{p'}u_{n'})\right]D^{1/2}_{p'n'},\end{aligned} \tag{14}$$

where F and G are the usual particle-hole (PH) and particle-particle (PP) coupled two-particle matrix elements.

The QRPA equations are recovered from (13) and (14) by taking $D_{pn}=1$. Within the RQRPA one first solves (5) in the quasiboson approximation [9]. The RQRPA ground state then reads

$$|0\rangle=N_0e^{\mathcal{S}}|BCS\rangle, \tag{15}$$

with

$$\mathcal{S}=\frac{1}{2}\sum_{pnp'n'J}\hat{J}^{-1}\left[\mathsf{C}_{pnp'n'}(J)\mathcal{A}^{\dagger}_{pn}(J)\mathcal{A}^{\dagger}_{p'n'}(J)\right]^{0}. \tag{16}$$

From (5) it turns out that the matrix C is the solution of

$$\sum_{pn}\mathsf{X}^{*}_{pn}(\lambda J)\mathsf{C}_{pnp'n'}(J)=\mathsf{Y}^{*}_{p'n'}(\lambda J),\ \text{for all } \lambda,\ J. \tag{17}$$

Finally, by making use of this equation one finds the quasiparticle occupations

$$\mathcal{N}_p=\sum_{\lambda Jn'}\hat{J}^2\hat{j}_p^{-2}|\mathsf{Y}_{pn'}(\lambda J)|^2;\ \ \mathcal{N}_n=\sum_{\lambda Jp'}\hat{J}^2\hat{j}_n^{-2}|\mathsf{Y}_{p'n}(\lambda J)|^2. \tag{18}$$

The value of D_{pn} follows from (7) and (18).

To evaluate the transition matrix elements for the $\beta^{\mp}$ decays

$$\langle \lambda J||\mathcal{O}(J;\pm)||0\rangle = \langle 0|\left[\Omega(\lambda\bar{J}),\mathcal{O}(J;\pm)\right]^0|0\rangle, \tag{19}$$

with

$$\mathcal{O}(J;\pm) = \sum_i O(J;i)t_{\pm}(i), \tag{20}$$

we only need their two quasiparticle components

$$\mathcal{O}(J;\pm) \doteq \sum_{pn}\left[\Lambda^0_{pn}(J;\pm)A^{\dagger}_{pn}(J) + (-)^J\Lambda^{0*}_{pn}(J;\mp)A_{pn}(\bar{J})\right], \tag{21}$$

where

$$\begin{aligned}\Lambda^0_{pn}(J;+) &= -\hat{J}^{-1}u_pv_n\langle p||O(J)||n\rangle,\\ \Lambda^0_{pn}(J;-) &= (-)^J\hat{J}^{-1}u_nv_p\langle p||O(J)||n\rangle^*.\end{aligned} \tag{22}$$

From (3) and (22) one gets

$$\langle \lambda J||\mathcal{O}(J;\pm)||0\rangle = \hat{J}\sum_{pn}\left[\Lambda^0_{pn}(J;\pm)\mathsf{X}^*_{pn}(\lambda J) + (-)^J\Lambda^{0*}_{pn}(J;\mp)\mathsf{Y}^*_{pn}(\lambda J)\right]D^{1/2}_{pn}.$$

The corresponding total strengths are

$$S(J;\pm) = \hat{J}^{-2}\sum_{\lambda}|\langle \lambda J||\mathcal{O}(J;\pm)||0\rangle|^2. \tag{23}$$

Within the RQRPA the BCS equations have to be solved subject to the condition that $|0\rangle$ has on the average the correct number of particles. This requirement gives

$$\mathsf{N}_t = \sum_t \hat{j}^2_t[v^2_t + (1-2v^2_t)\mathcal{N}_t], \tag{24}$$

N_p and N_n being the number of active protons and neutrons in solving the gap equations.

We conclude the presentation of the formalism by noting that: (a) when the factors D_{pn}, which are functions of the amplitudes Y are substituted into the renormalized matrices A and B, (12) becomes a nonlinear system of coupled equations for the X and Y amplitudes; and (b) these equations have to be solved self-consistently together with the new BCS conditions (24). This is the price to be paid in order to take into account the GSC within the QRPA problem in an appropriate way.

We will resort now to the simplest version of the QRPA for the $\beta\beta$-decay, called the single mode model (SMM), in which a single RPA equation is solved with two BCS vacua

[12], and only one intermediate state $J^\pi = 1^+$ enters into the play [4]. Eqs. (14) read in this case

$$
\begin{aligned}
A_{pn} &\equiv \omega_0 + \rho_p\rho_n\left[(u_p^2 v_n^2 + \bar{v}_p^2\bar{u}_n^2)F(pn;1) + (u_p^2\bar{u}_n^2 + \bar{v}_p^2 v_n^2)G(pn;1)\right] D_{pn},\\
B_{pn} &\equiv 2\rho_p\rho_n\bar{v}_p\bar{u}_n v_n u_p[F(pn;1) - G(pn;1)]D_{pn},
\end{aligned}
$$

where $\omega_0 = -[G(pp;0) + G(nn;0)]/4$ is the unperturbed energy. The unbarred (barred) quantities indicate that the quasiparticles are defined with respect to the initial (final) nucleus; $\rho_p^{-1} = u_p^2 + \bar{v}_p^2$, $\rho_n^{-1} = \bar{u}_n^2 + v_n^2$. All the remaining notation is self explanatory. The perturbed energy and D_{pn} are obtained by solving self-consistently the set of equations:

$$
\omega = \sqrt{A_{pn}^2 - B_{pn}^2}, \quad D_{pn} = 1 - f\frac{A_{pn} - \omega}{2\omega}, \quad v_t^2 = \frac{\mathsf{N}_t f - 3(1 - D_{pn})}{f\hat{j}_t^2 - 6(1 - D_{pn})},
$$

with $f \equiv 3(\hat{j}_p^{-2} + \hat{j}_n^{-2})$.

The transition $\beta\beta_{2\nu}$ matrix element is

$$
\mathcal{M}_{2\nu} = \mathcal{M}_{2\nu}^0 D_{pn}\left(\frac{\omega_0}{\omega}\right)^2\left(1 + \frac{G(pn;1)D_{pn}}{\omega_0}\right), \quad \mathcal{M}_{2\nu}^0 = \frac{\rho_p\rho_n\bar{v}_p\bar{u}_n v_n u_p}{\omega_0}|\langle p||\sigma||n\rangle|^2 \tag{25}
$$

with $\mathcal{M}_{2\nu}^0$ being the corresponding unperturbed (BCS) value.

Numerical calculations have been performed for the $^{100}Mo \rightarrow {}^{100}Ru$ system, where the appropriate intermediate state is $[0g_{7/2}(n)0g_{9/2}(p)]^1$, and $\mathsf{N}_p = 2$ and $\mathsf{N}_n = 2$ ($\mathsf{N}_p = 4$ and $\mathsf{N}_n = 0$) for the initial (final) state. We have used a δ-force (in units of $MeV fm^3$): $V = -4\pi(v_s P_s + v_t P_t)\delta(r)$, with different strength constants v_s and v_t for the PH, PP and pairing channels. Thus, instead of the parameter g^{pp} we use here the ratio $\mathrm{t} = v_t^{pp}/v_s^{pair}$, whose physical value is $\mathrm{t} \cong 1.5$. The remaining parameters for the SMM have been taken to be $v_s^{ph} = 55$, $v_t^{ph} = 92$ and $v_s^{pair} = 55$ [3]. The results obtained within the QRPA (dashed lines) and the RQRPA (solid lines) for ω and for $\mathcal{M}_{2\nu}$ are shown in Fig. 1. As expected, the QRPA collapses close to $\mathrm{t} = 1.5$. Contrarily, in the RQRPA the energy decreases asymptotically when $\mathrm{t} \rightarrow \infty$. For the sake of comparison, in the same figure, are also presented the results for the energy of the lowest $J^\pi = 1^+$ state and for the 2ν matrix element of a full QRPA calculation (dotted lines), as described in ref. [3]. This calculation, that involves an eleven dimensional model space, both for protons and neutrons, also collapses. (It is very gratifying that the simple formula (25) contains the main physics involved in such a relative sizable calculations.)

In summary, we have investigated the importance of GSC effects on the solutions of the EM for charge-exchange excitations in the renormalized QRPA. The SMM shows that, contrarily to what happens in the usual QRPA, the inclusion of the GSC in the EM avoids collapse for physical values of the PP coupling strength. However, the amplitude $\mathcal{M}_{2\nu}$

still passes through zero in the RQRPA, although at somewhat higher value of t (or g^{pp}). It is also evident that, in the QRPA, the physical mechanisms responsible for the zero and the collapse of $\mathcal{M}_{2\nu}$ are not the same. The behavior of this amplitude in the RQRPA is not anymore delineated by Eq. (1), and the dependence of the calculated $\beta\beta_{2\nu}$ transition rates on g^{pp} is weakened. In view of Eqs. (1) and (2), all that was just said for the 2ν mode can be extrapolated also to the 0ν mode. It is well known that the contributions of intermediate states with $J^\pi \neq 1$ are quite sizeable in the neutrinoless decay for physical value of $g^{pp} \cong 1$, where it is very likely that the $\mathcal{M}_{0\nu}(J^\pi = 1^+)$ goes to zero even in the RQRPA case. But as the dynamical calculation does not collapse any more, we could now have more confidence in establishing the upper limit for the neutrino mass. Thus, the effect of the GSC in the EM appear in this context as particularly relevant. We have also found that a full RQRPA calculation for the $\mathcal{M}_{2\nu}$ amplitude agrees qualitatively with the SMM estimate. But, in analyzing the Ikeda sum rule $S(J^\pi = 1^+; +) - S(J^\pi = 1^+; -) = N - Z$, we discovered that it is not fulfilled within the RQRPA. In fact, the deviations from this condition grow as the GSC increase (or as the PP strength parameter increases). On the other hand, we have verified numerically that the similar requisite for the Fermi transitions is fulfilled in our formalism, when only the states $J^\pi = 0^+$ are considered in the Eqs. (18). It should be stressed that the constraints (24) plays a crucial role regarding this point. When the usual BCS constraint on the number of particles is used [10], the sum rule for the Fermi transitions is never fulfilled. That the Ikeda sum rule is necessarily violated in the RQRPA, when the usual BCS occupation numbers are employed, is seen immediately from the relation

$$S(J^\pi = 1^+; +) - S(J^\pi = 1^+; -) = \frac{1}{3} \sum_{pn} | < p||\sigma||n > |^2 (v_n^2 - v_p^2) D_{pn},$$

which yields $N - Z$ only when $D_{pn} \equiv 1$. Why the Ikeda sum rule is not satisfied, even when the condition (24) is adopted, is still an open question. In summary, we feel that, before a quantitative comparison of the calculations with the experimental data could be done, the behavior of the sum rules in the RQRPA should be thoroughly elucidated and this is our next goal.

Finally, it should be mentioned that, after our work has been completed, we have learned that a similar study has been performed by Toivanen and Suhonen [13].

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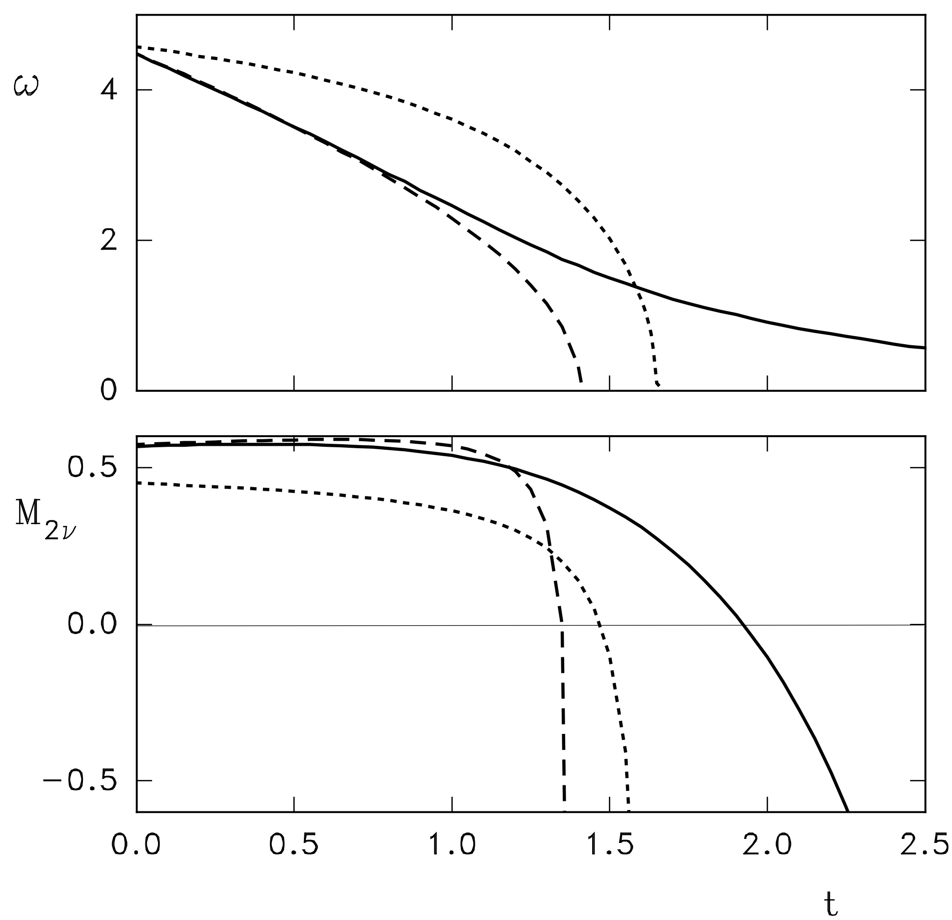
Figure Captions

Figure 1: Energies ω (in MeV) of the lowest 1^+ state of ^{100}Tc and the matrix elements $\mathcal{M}_{2\nu}$ (in $[MeV]^{-1}$) for the $^{100}Mo \rightarrow {}^{100}Ru$ system. The single mode model results are indicated by the dashed lines for the QRPA and by the solid lines for the RQRPA. The results of a full QRPA calculation [3] are represented by dotted lines.



ω
4
2
0
$M_{2\nu}$
0.5
0.0
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
t